

Evolving hypersurfaces by their mean curvature in the background manifold evolving by Ricci flow*

Weimin Sheng and Haobin Yu

Abstract

We consider the problem of deforming a one-parameter family of hypersurfaces immersed into closed Riemannian manifolds with positive curvature operator. The hypersurface in this family satisfies mean curvature flow while the ambient metric satisfying the normalized Ricci flow. We prove that if the initial metric of the background manifold is sufficiently pinched and the initial hypersurface also satisfies a suitable pinching condition, then either the hypersurfaces shrink to a round point in finite time or converge to a totally geodesic sphere as the time tends to infinity.

Keywords: mean curvature flow, normalized Ricci flow, totally geodesic sphere

1 Introduction

Let (N^{n+1}, \bar{g}) be a complete, simply connected Riemannian manifold, $X(\cdot, t): M^n \rightarrow N^{n+1}$ be a one-parameter family of smooth oriented hypersurface immersions, satisfying the evolution equation

$$\begin{cases} \frac{\partial X(x, t)}{\partial t} = -H(x, t)\nu(x, t), x \in M^n, t > 0 \\ X(\cdot, t) = X_0, \end{cases} \quad (1.1)$$

where $H(x, t)$ is the mean curvature of the hypersurface $X(\cdot, t)$ at the point $X(x, t)$, $\nu(x, t)$ is the outer unit normal to $X(\cdot, t)$ and X_0 is a given oriented hypersurface in N^{n+1} . This is the well-known mean curvature flow which has been studied extensively, when the background is a fixed Riemannian manifold, see [3, 7, 9, 11, 15, 17] for instance.

In [11], Huisken got an important monotonicity formula for hypersurfaces in the Gaussian shrinker background. So it is reasonable to consider the mean curvature flow in a moving ambient space. In particular, when the metric of N^{n+1} satisfies the Ricci flow, we call the coupled evolutions as the "Ricci-Mean curvature flow". Magni-Mantegazza-Tsatis[13] showed a similar monotonicity as Huisken's for mean curvature flow in a gradient Ricci soliton background. Recently, John lott [12] presented a very valuable explanation on the "Ricci-Mean curvature flow". He used the variation method to get the evolution equations of the second fundamental form and the mean curvature. In the case of N^{n+1} being a gradient Ricci soliton, he introduced the concept of mean curvature soliton which

*This research was supported by Natural Science Foundation of China, Grant No. 11131007, and Zhejiang Provincial Natural Science Foundation of China, Grant No. LY14A010019.

can be regarded as the generalization of self-shrinker. In [5], Han and Li studied a surface immersed in a Kähler surface evolved by its mean curvature flow while the Kähler surface evolved by Kähler-Ricci flow. They proved if the Kähler surface is sufficiently close to a Kähler-Einstein surface and the initial surface is sufficiently close to a holomorphic curve, then the surface converges to a holomorphic curve along the Kähler-Ricci mean curvature flow. This is the first convergence result on Ricci-Mean curvature flow.

In this paper, we consider a one-parameter family of immersions $X(\cdot, t) : M^n \rightarrow (N^{n+1}, \bar{g}(t))$, which satisfies

$$\begin{cases} \frac{\partial X(x, t)}{\partial t} = -H(x, t)\nu(x, t), & x \in M^n, t > 0 \\ \frac{\partial \bar{g}(t)}{\partial t} = -2\bar{Ric}(t) + \frac{2\bar{r}}{n+1}\bar{g}(t), & \bar{g}(0) = \bar{g}_0 \end{cases} \quad (1.2)$$

where \bar{r} is the average of the scalar curvature of the background metric \bar{g} . In [10], Huisken considered the deformation of hypersurfaces of the sphere by their mean curvature, he proved if the initial hypersurface satisfies a suitable pinching condition, then either the hypersurfaces shrink to a round point in finite time or the equation has a smooth solution M_t for $0 \leq t < \infty$ and M_t converges to a totally geodesic hypersurface when t tends to ∞ . We can show the similar result also holds under (1.2), under the assumption that the metric \bar{g}_0 of N^{n+1} has positive curvature operator and is sufficiently pinched. To be precise, we prove

Theorem 1.1 *There exists a positive constant $\varepsilon_0 \leq \frac{1}{4(n+1)}$ small, such that if (N^{n+1}, \bar{g}_0) satisfies*

$$\|\bar{R}_{\alpha\beta\gamma\delta} - (\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma})\|^2 \leq \varepsilon_0^2, \quad \|\bar{\nabla}\bar{R}m\| \leq \varepsilon_0 \quad (1.3)$$

where the norm $\|\cdot\|$ is taken with respect to \bar{g}_0 , and the initial hypersurface M_0 immersed into (N^{n+1}, \bar{g}_0) satisfies

$$\|A\|^2 \leq \alpha_n H^2 + 1 \quad (1.4)$$

with

$$\alpha_2 = \frac{11}{16}, \quad \alpha_n = \frac{4}{4n-3}, n \geq 3$$

then for the solution to (1.2), either

- (1) M_t shrink to a round point in finite time $T < \infty$, and $\max_{M_t} |H| \rightarrow \infty$ as $t \rightarrow T$; or
- (2) the equation has a solution M_t for $0 \leq t < \infty$, and M_t converge to a totally geodesic sphere in C^∞ -topology.

From (1.3), we know (N^{n+1}, \bar{g}_0) has positive curvature, by the result of Hamilton[4] and Huisken[8], $(N^{n+1}, \bar{g}(t))$ converge to the spherical space form as $t \rightarrow \infty$. But it is not easy to see the behaviour of the hypersurface with its induced metric evolving under the mean curvature flow. The key problem is when the mean curvature flow will develop singularities in a finite time. If it will not develop a singularity, we wish to understand which one is faster between the background manifold to the sphere under Ricci flow and the immersed hypersurface to its totally geodesic hypersurface under mean curvature flow.

The rest of the paper is organized as follows. In section 2, we give some preliminary and get the evolution equations for quantities of hypersurfaces. In section 3, we derive a pinching estimate to control the second fundamental form by using an inequality derived above. In section 4, we show the gradient of the mean curvature can be controlled by the mean curvature itself. We give the proof of Theorem 1.1 in the last section .

2 Preliminaries and Evolution Equations

In this section, we gather some estimates which will be used later. We choose a local frames field $\{e_0, e_1, \dots, e_n\}$ in N^{n+1} such that $e_0 = \nu, e_i = \frac{\partial X}{\partial x_i}$ on $X(\cdot)$. Let ∇ and Δ denote the connection and Laplacian on M determined by the induced metric g . We denote all the quantities on (N^{n+1}, \bar{g}) with a bar, for example, by $\bar{\nabla}$ the covariant derivative, $\bar{\Delta}$ the Laplacian, and $\bar{R}m = \bar{R}_{\alpha\beta\gamma\delta}$ the Riemannian curvature tensor. Let $\overset{\circ}{R}m$ be the tracefree part of curvature operator, i.e,

$$\overset{\circ}{R}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - \frac{R}{n(n+1)}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma})$$

and

$$\bar{E}_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\beta\gamma\delta} - \frac{\bar{r}}{n(n+1)}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma})$$

We will show the exponential decay of $\|\bar{E}\|$ and $\|\bar{\nabla}\bar{R}m\|$ under the normalized Ricci flow. First we consider the Ricci flow with $\tilde{g}(\cdot, 0) = \bar{g}_0$,

$$\frac{\partial}{\partial t}\tilde{g}_{\alpha\beta} = -2\tilde{R}_{\alpha\beta}, \quad \tilde{t} \in [0, T),$$

where T is the singular time, and (N^{n+1}, \bar{g}_0) satisfies the assumption (1.3) for some constant ε_0 .

By our assumption, the sectional curvature $\tilde{K}(x, 0)$ and the scalar curvature $\tilde{R}(x, 0)$ of \bar{g}_0 satisfy

$$1 - \varepsilon_0 \leq \tilde{K}(x, 0) \leq 1 + \varepsilon_0, \quad n(n+1)(1 - \varepsilon_0) \leq \tilde{R}(x, 0) \leq n(n+1)(1 + \varepsilon_0), \quad (2.1)$$

which is followed by

$$\begin{aligned} \|\overset{\circ}{R}m\|^2(x, 0) &\leq \|\tilde{R}_{\alpha\beta\gamma\delta} - (\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma})\|^2(x, 0) + \|(1 - \frac{\tilde{R}}{n(n+1)})(\tilde{g}_{\alpha\gamma}\tilde{g}_{\beta\delta} - \tilde{g}_{\alpha\delta}\tilde{g}_{\beta\gamma})\|^2(x, 0) \\ &\leq \varepsilon_0^2 + 2n(n+1)\varepsilon_0^2 \leq \frac{2(n+1)^2\varepsilon_0^2\tilde{R}^2(x, 0)}{n^2(n+1)^2(1 - \varepsilon_0)^2} \\ &\leq \frac{\tilde{R}^2(x, 0)}{4n^2(n+1)^2} \end{aligned}$$

We need the following results which were derived by Huisken in [8] and take the following version in our case.

Lemma 2.1 (*Theorem 3.1 of [8]*). *Under the assumption (1.3), it always holds*

$$\|\tilde{R}m\|^2 - \frac{2}{n(n+1)}\tilde{R}^2 \leq \frac{\tilde{R}^2}{4n^2(n+1)^2},$$

which implies the sectional curvature $\tilde{K}(x, \tilde{t})$ of (N^{n+1}, \tilde{g}) satisfies $\tilde{K}(x, \tilde{t}) \geq \frac{\tilde{R}(x, \tilde{t})}{2n(n+1)}$. Moreover, there exist constants $C_0 < \infty$ and $\delta_0 \in (0, 1)$ depending only on n such that $\|\overset{\circ}{R}m\|^2 \leq C_0\tilde{R}^{2-\delta_0}$ holds on $0 \leq \tilde{t} < T$.

Remark 2.1 In Theorem 3.1 of [8], Huisken gave the explicit expression of C_0 , i.e.,

$$C_0 = \sup_{(N^{n+1}, \bar{g}(0))} \|\overset{\circ}{\tilde{R}m}\|^2 \tilde{R}^{\delta_0-2},$$

by our assumption (1.3), $C_0 \leq \varepsilon_0^2$.

Lemma 2.2 (Theorem 4.1 of [8]). For any $\eta > 0$, we can find $C(\eta)$ depending only on η and n , such that on $0 \leq \tilde{t} < T$ we have

$$\|\tilde{\nabla} \tilde{R}\|^2 \leq \eta \tilde{R}^3 + C(\eta)$$

Let V be the volume of (N^{n+1}, \bar{g}_0) . We choose the normalization factor $\psi(\tilde{t}) = (\frac{\int d\mu_{\bar{g}}}{V})^{-\frac{2}{n+1}}$ and a new time scale $t = \int_0^{\tilde{t}} \psi(s) ds$, then $\bar{g}(t) = \psi(\tilde{t}) \bar{g}(\tilde{t})$ satisfy the normalized Ricci flow with $\bar{g}(0) = \bar{g}_0$ and $\frac{d \ln \psi}{dt} = \frac{2}{n+1} \bar{r}$. Define a function φ by $\varphi(t) = \psi(\tilde{t})$. The following evolution equations for the normalized Ricci flow were established by Hamilton in [4].

Lemma 2.3 Under the normalized Ricci flow,

$$\begin{aligned} (1) \quad & \frac{\partial}{\partial t} \|\bar{R}m\|^2 = \bar{\Delta} \|\bar{R}m\|^2 - 2 \|\bar{\nabla} \bar{R}m\|^2 + 4 \bar{Q}_{\alpha\beta\gamma\delta} \bar{R}_{\alpha\beta\gamma\delta} - \frac{4}{n+1} \bar{r} \|\bar{R}m\|^2, \\ (2) \quad & \frac{\partial}{\partial t} \|\bar{Ric}\|^2 = \bar{\Delta} \|\bar{Ric}\|^2 - 2 \|\bar{\nabla} \bar{Ric}\|^2 + 4 \bar{R}_{\alpha\beta} \bar{R}_{\gamma\delta} \bar{R}_{\alpha\gamma\beta\delta} - \frac{4}{n+1} \bar{r} \|\bar{Ric}\|^2, \\ (3) \quad & \frac{\partial}{\partial t} \bar{R} = \bar{\Delta} \bar{R} + 2 \|\bar{Ric}\|^2 - \frac{2}{n+1} \bar{r} \bar{R}. \end{aligned}$$

where $\bar{Q}_{\alpha\beta\gamma\delta} = (\bar{B}_{\alpha\beta\gamma\delta} - \bar{B}_{\alpha\beta\delta\gamma} - \bar{B}_{\alpha\delta\beta\gamma} + \bar{B}_{\alpha\gamma\beta\delta}) \bar{R}_{\alpha\beta\gamma\delta}$, and $\bar{B}_{\alpha\beta\gamma\delta} = \bar{R}_{\alpha\eta\beta\theta} \bar{R}_{\gamma\eta\delta\theta}$.

Now we are ready to prove

Theorem 2.1 There exist some universal constant \bar{C} and λ depending only on n such that under the normalized Ricci flow,

$$\|\bar{E}\|(\cdot, t) \leq \bar{C} \varepsilon_0 e^{-\lambda t}, \quad \|\bar{\nabla} \bar{R}m(\cdot, t)\| \leq \bar{C} \varepsilon_0 e^{-\lambda t}$$

Proof. By the evolution equations,

$$\begin{aligned} \frac{d\bar{r}}{dt} &= \frac{\int \frac{\partial \bar{R}}{\partial t} d\mu}{\int d\mu} = 2 \left(\frac{\int \|\bar{Ric}\|^2 d\mu}{\int d\mu} - \frac{\bar{r}^2}{n+1} \right) \\ &\geq \frac{2}{n+1} \left(\frac{\int \bar{R}^2 d\mu}{\int d\mu} - \bar{r}^2 \right) \geq 0 \end{aligned}$$

so

$$n(n+1)(1 - \varepsilon_0) \leq \bar{r}(0) \leq \bar{r}(t), \quad t \in [0, \infty) \quad (2.2)$$

Using the upper bound for the sectional curvature of (N^{n+1}, \bar{g}_0) and Klingenberg's Lemma (Theorem 5.10 of [1]), the injectivity radius r_0 of (N^{n+1}, \bar{g}_0) satisfies $r_0 \geq \frac{\pi}{\sqrt{1+\varepsilon_0}}$. Let ω_{n+1} be the volume of unit sphere S^{n+1} . Then the volume comparison theorem implies $V \geq \omega_{n+1}(1 + \varepsilon_0)^{-\frac{n+1}{2}}$. Since $(N^{n+1}, \bar{g}(t))$ converges to $(N^{n+1}, \bar{g}_\infty)$ with constant curvature K_∞ , thus

$$\omega_{n+1}(1 + \varepsilon_0)^{-\frac{n+1}{2}} \leq V = V_\infty = \omega_{n+1} K_\infty^{-\frac{n+1}{2}},$$

which follows by

$$\bar{r}(t) \leq \bar{r}_\infty = n(n+1)K_\infty \leq n(n+1)(1+\varepsilon_0) \quad (2.3)$$

As

$$\frac{d \ln \varphi(t)}{dt} = \frac{d \ln \psi(\tilde{t})}{dt} = \frac{2}{n+1} \bar{r}(t) \geq 2,$$

we have

$$\varphi(t) \geq \varphi(0)e^{2t} = \psi(0)e^{2t}$$

By Lemma 2.2,

$$\|\bar{\nabla} \bar{R}\| \leq \eta \bar{R}^{\frac{3}{2}} + C(\eta) \varphi(t)^{-\frac{3}{2}} \leq \eta \bar{R}^{\frac{3}{2}} + C(\eta)$$

Step 1. We first show there exists a constant C_n depending only on n such that for any initial metric \bar{g}_0 satisfying (1.3), the corresponding normalized Ricci flow $(N^{n+1}, \bar{g}(t))$ satisfies

$$\bar{R}(x, t) \leq C_n, \quad \forall (x, t) \in N^{n+1} \times [0, \infty) \quad (2.4)$$

We show this by a contradiction argument. Suppose not, then there exist a sequence of metrics \bar{g}_k satisfying (1.3), $x_k \in N^{n+1}$ and $t_k > 0$ such that $A_k = \bar{R}_k(x_k, t_k) \rightarrow \infty$ as $k \rightarrow \infty$. For any $\eta > 0$, there exists an integer k , such that the metric $\bar{g}_k(t)$ satisfies

$$\|\bar{\nabla} \bar{R}\|(\cdot, t_k) \leq 2\eta A_k^{\frac{3}{2}}$$

Now for any point y with $d_{\bar{g}_k(t_k)}(y, x_k) \leq \frac{1}{\sqrt{\eta A_k}}$, we have

$$\bar{R}_k(y) \geq A_k - 2d_{\bar{g}_k(t_k)}(y, x_k)\eta A_k^{\frac{3}{2}} \geq (1 - 2\sqrt{\eta})A_k$$

Then by Lemma 2.1, the sectional curvature $\bar{K}(y, t_k)$ of $(N^{n+1}, \bar{g}_k(t_k))$ satisfies

$$\bar{K}(y, t_k) \geq \frac{1 - 2\sqrt{\eta}}{2n(n+1)} A_k$$

On the other hand, by Myers' theorem, any geodesic from x_k with length larger than $\frac{2(n+1)\pi}{\sqrt{(1-2\sqrt{\eta})A_k}}$ must have conjugate points. Thus by choosing $\eta < \frac{1}{8(n+1)^2\pi}$ and k large enough, for any $x \in (N^{n+1}, \bar{g}_k(t_k))$, we have

$$\bar{K}(x, t_k) \geq \frac{1 - 2\sqrt{\eta}}{2n(n+1)} A_k.$$

Hence $\text{Vol}(N^{n+1}, \bar{g}_k(t_k)) \rightarrow 0$ as $k \rightarrow \infty$, which contradicts with the fact that $(N^{n+1}, \bar{g}_k(t))$ has constant volume $V \geq \omega_{n+1}(1+\varepsilon_0)^{-\frac{n+1}{2}}$.

Step 2. We next show the exponentially decreasing of $\|\bar{\nabla} \bar{R}m\|$ under the normalized Ricci flow.

Let C_n denote the universal constants depending only on n . By Lemma 2.1 and (2.4),

$$\|\overset{\circ}{\bar{R}m}\|^2 = \|\bar{R}m\|^2 - \frac{2\bar{R}^2}{n(n+1)} \leq C_0 \bar{R}^{2-\delta_0} \varphi(t)^{-\delta_0} \leq C_n \varepsilon_0^2 e^{-2\delta_0 t} \quad (2.5)$$

Let $f = \|\bar{R}m\|^2 - \frac{2\bar{R}^2}{n(n+1)}$. By Lemma 2.3,

$$\frac{\partial}{\partial t} f \leq \bar{\Delta} f - 2\|\bar{\nabla} \bar{R}m\|^2 + \frac{4\|\bar{\nabla} \bar{R}\|^2}{n(n+1)} + 4\bar{Q}_{\alpha\beta\gamma\delta} \bar{R}_{\alpha\beta\gamma\delta} - \frac{8\bar{R}}{n(n+1)} \|\bar{Ric}\|^2, \quad (2.6)$$

By

$$\begin{aligned}\bar{Q}_{\alpha\beta\gamma\delta}\bar{R}_{\alpha\beta\gamma\delta} &\leq \|\bar{Q}_{\alpha\beta\gamma\delta}(\bar{R}_{\alpha\beta\gamma\delta} - \frac{\bar{R}}{n(n+1)}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}))\| \\ &\quad + \frac{\bar{R}}{n(n+1)}\bar{Q}_{\alpha\beta\gamma\delta}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma})\end{aligned}\quad (2.7)$$

An easy calculation shows

$$\bar{Q}_{\alpha\beta\gamma\delta}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma}) = 2(\|\bar{R}m\|^2 + \|\bar{Ric}\|^2) - 4\bar{R}_{\alpha\beta\gamma\theta}\bar{R}_{\gamma\beta\alpha\theta} \quad (2.8)$$

By taking C_n large enough, we have

$$\begin{aligned}\bar{R}_{\alpha\beta\gamma\theta}\bar{R}_{\gamma\beta\alpha\theta} &\geq \frac{\bar{R}}{n(n+1)}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\theta} - \bar{g}_{\alpha\theta}\bar{g}_{\beta\gamma})\bar{R}_{\gamma\beta\alpha\theta} \\ &\quad - \|\frac{\bar{R}}{n(n+1)}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\theta} - \bar{g}_{\alpha\theta}\bar{g}_{\beta\gamma}) - \bar{R}_{\alpha\beta\gamma\theta}\|\|\bar{R}_{\gamma\beta\alpha\theta}\| \\ &\geq \frac{\bar{R}^2}{n(n+1)} - C_n\varepsilon_0 e^{-\delta_0 t}\end{aligned}\quad (2.9)$$

Substituting (2.8) and (2.9) into (2.7) gives

$$\bar{Q}_{\alpha\beta\gamma\delta}\bar{R}_{\alpha\beta\gamma\delta} \leq \frac{2\bar{R}\|\bar{Ric}\|^2}{n(n+1)} + \frac{2\bar{R}}{n(n+1)}(\|\bar{R}m\|^2 - \frac{2\bar{R}^2}{n(n+1)}) + C_n\varepsilon_0 e^{-\delta_0 t} \quad (2.10)$$

Combining (2.6) and (2.10), we get

$$\frac{\partial}{\partial t}f \leq \bar{\Delta}f - 2\|\bar{\nabla}\bar{R}m\|^2 + \frac{4\|\bar{\nabla}\bar{R}\|^2}{n(n+1)} + C_n\varepsilon_0 e^{-\delta_0 t} \quad (2.11)$$

By Lemma 4.3 in [8],

$$\|\bar{\nabla}\bar{Ric}\|^2 - \frac{\|\bar{\nabla}\bar{R}\|^2}{n+1} \geq (\frac{3n+1}{2n(n+3)} - \frac{1}{n+1})\|\bar{\nabla}\bar{R}\|^2 = \frac{(n-1)^2}{2n(n+1)(n+3)}\|\bar{\nabla}\bar{R}\|^2$$

Using Lemma 2.3, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\|\bar{Ric}\|^2 - \frac{\|\bar{R}\|^2}{n+1}) &\leq \bar{\Delta}(\|\bar{Ric}\|^2 - \frac{\|\bar{R}\|^2}{n+1}) - 2(\|\bar{\nabla}\bar{Ric}\|^2 - \frac{\|\bar{\nabla}\bar{R}\|^2}{n+1}) \\ &\quad + 4(\bar{R}_{\alpha\beta} - \frac{\bar{R}}{n+1}\bar{g}_{\alpha\beta})\bar{R}_{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta} \\ &\leq \bar{\Delta}(\|\bar{Ric}\|^2 - \frac{\|\bar{R}\|^2}{n+1}) - \frac{(n-1)^2}{n(n+1)(n+3)}\|\bar{\nabla}\bar{R}\|^2 + C_n\varepsilon_0^2 e^{-\delta_0 t},\end{aligned}\quad (2.12)$$

where we have used the fact

$$(\bar{R}_{\alpha\beta} - \frac{\bar{R}}{n+1}\bar{g}_{\alpha\beta})\bar{R}_{\gamma\delta}\bar{R}_{\alpha\gamma\beta\delta} \leq \bar{R}(\|\bar{Ric}\|^2 - \frac{\bar{R}^2}{n+1}) \leq C_n\varepsilon_0^2 e^{-\delta_0 t}$$

In additional,

$$\frac{\partial}{\partial t}\|\bar{\nabla}\bar{R}m\|^2 \leq \bar{\Delta}\|\bar{\nabla}\bar{R}m\|^2 - 2\|\bar{\nabla}^2\bar{R}m\|^2 + C_n\|\bar{\nabla}\bar{R}m\|^2 \quad (2.13)$$

Now let

$$F = \|\bar{\nabla} \bar{R}m\|^2 + C_n f + C_n^2 (\|\bar{Ric}\|^2 - \frac{\|\bar{R}\|^2}{n+1})$$

Combining (2.11), (2.12) and (2.13) gives

$$\frac{\partial F}{\partial t} \leq \bar{\Delta} F - C_n F + 2C_n^3 \varepsilon_0 e^{-\delta_0 t}$$

Since $F(\cdot, 0) \leq 3C_n^3 \varepsilon_0^2$, the standard maximum principle implies that there exists a constant λ_1 depending only on n such that

$$\|\bar{\nabla} \bar{R}m\|^2 \leq C \varepsilon_0^2 e^{-2\lambda_1 t}. \quad (2.14)$$

Step 3. We want to get an uniformly upper bound for the diameter of $(N^{n+1}, \bar{g}(t))$ under the normalized Ricci flow.

Consider Perelman's \mathcal{W} -functional [14],

$$\mathcal{W}(\tilde{g}, f, \tau) = \int_{N^{n+1}} [\tau(\tilde{R} + \|\tilde{\nabla} f\|^2) + f - (n+1)] (4\pi\tau)^{-\frac{n+1}{2}} e^{-f} d\mu$$

where f is a smooth function on N^{n+1} , and τ is a positive scale parameter. Let

$$\rho = (4\pi\tau)^{-\frac{n+1}{4}} e^{-\frac{f}{2}}$$

Now we set

$$\mu(\tilde{g}, \tau) = \inf \{ \mathcal{W}(\tilde{g}, f, \tau) \mid \rho \in C^\infty(N^{n+1}), \int_{N^{n+1}} \rho^2 d\mu = 1 \}$$

By our assumption for the initial metric \bar{g}_0 and the Theorem A in [18],

$$\mu(\tilde{g}(0), \tau) \geq -CT - C, \quad \tau \in (0, 2T]$$

where C is a constant depending only on n , and $T < \frac{n+1}{2\bar{R}_{\min}(0)}$ is the maximal existence time for the (unnormalized) Ricci flow. Thus

$$\mu(\tilde{g}(0), \tau) \geq -C_n, \quad \forall \tau \in (0, 2T]$$

Now combining the upper bound for the scalar curvature of $\bar{g}(\cdot, t)$, Perelman's no local collapsing theorem I' [14], and a local injectivity radius estimate of Cheeger-Gromov-Taylor [2], we can get the following

Proposition 2.1 *There exists a constant $c_n > 0$ depending on n , such that*

$$\text{inj}(N^{n+1}, \bar{g}(t)) \geq c_n, \quad \forall t \in [0, \infty)$$

As $(N^{n+1}, \bar{g}(t))$ has constant volume, it follows that diameter of $(N^{n+1}, \bar{g}(t))$ has a uniformly upper bound

$$\text{diam}(N^{n+1}, \bar{g}(t)) \leq C_n, \quad \forall t \in [0, \infty) \quad (2.15)$$

Combining (2.5), (2.14) and (2.15) yields the desired estimate. ■

Remark 2.2 *Once getting the exponential decay of $\|\bar{R}m\|^2$ and $\|\bar{\nabla} \bar{R}m\|^2$, one can show $\|\bar{\nabla}^k \bar{R}m\|^2$ are also exponentially decreasing, see [4] for details.*

Remark 2.3 Note that from (2.2) and (2.3), we have derived the uniform bound for $\bar{r}(t)$,

$$n(n+1)(1-\varepsilon_0) \leq \bar{r}(t) \leq n(n+1)(1+\varepsilon_0), \quad t \in [0, \infty) \quad (2.16)$$

We denote by $S = \{S_i\}$ the vector with components $S_i = \bar{R}_{0i}$, the following estimate was derived by Huisken in [9].

Lemma 2.4 For any $\eta > 0$,

$$\|\nabla A\|^2 \geq \left(\frac{3}{n+2} - \eta\right) \|\nabla H\|^2 - \frac{2}{n+2} \left(\frac{2}{n+2} \eta^{-1} - \frac{n}{n-1}\right) \|S\|^2$$

By a direct calculation or using the results in [12], we could establish the following evolution equations

Lemma 2.5

$$\begin{aligned} (1) \quad & \frac{\partial g_{ij}}{\partial t} = -2Hh_{ij} - 2\bar{R}_{ij} + \frac{2}{n+1} \bar{r}g_{ij}, \\ (2) \quad & \frac{\partial h_{ij}}{\partial t} = \Delta h_{ij} - 2Hh_{ip}h_{jp} + |A|^2 h_{ij} + \frac{\bar{r}}{n+1} h_{ij} + P_{ij} - \bar{\nabla}_0 \bar{R}_{0i0j}, \\ (3) \quad & \frac{\partial H}{\partial t} = \Delta H + \|A\|^2 H + 2\bar{R}_{ij}h_{ij} - \frac{\bar{r}H}{n+1} - \bar{\nabla}_0 \bar{R}_{00}, \\ (4) \quad & \frac{\partial \|A\|^2}{\partial t} = \Delta \|A\|^2 - 2\|\nabla A\|^2 + 2\|A\|^4 + 2P_{ij}h_{ij} + 4\bar{R}_{ij}h_{ik}h_{jk} - \frac{2\bar{r}\|A\|^2}{n+1} - 2\bar{\nabla}_0 \bar{R}_{0i0j}h_{ij}. \end{aligned}$$

Here $P_{ij} = 2h_{kl}\bar{R}_{kilj} - h_{il}\bar{R}_{jklk} - h_{jl}\bar{R}_{iklk}$.

For simplicity, we will use the following denotation throughout the paper,

$$u = 2\bar{R}_{ij}h_{ij} - \frac{\bar{r}H}{n+1} - \bar{\nabla}_0 \bar{R}_{00}$$

$$v = 2P_{ij}h_{ij} + 4\bar{R}_{ij}h_{ik}h_{jk} - \frac{2\bar{r}\|A\|^2}{n+1} - 2\bar{\nabla}_0 \bar{R}_{0i0j}h_{ij}$$

Now we choose ε_0 small, such that

$$\varepsilon_1 = (\bar{C} + 1)\varepsilon_0 \leq \frac{1}{2^7 n} \quad (2.17)$$

By

$$\|\bar{E}\| = \|\bar{R}_{\alpha\beta\gamma\delta} - \frac{\bar{r}}{n(n+1)}(\bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta} - \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma})\| \leq \bar{C}\varepsilon_0 e^{-\lambda t}, \quad (2.18)$$

it follows the sectional curvature $\bar{K}(x, t)$ satisfies

$$\frac{\bar{r}}{n(n+1)} - \bar{C}\varepsilon_0 e^{-\lambda t} \leq \bar{K}(x, t) \leq \frac{\bar{r}}{n(n+1)} + \bar{C}\varepsilon_0 e^{-\lambda t}$$

Taking the trace on β and δ in (2.18) gives

$$\|\bar{R}_{ij} - \frac{\bar{r}}{n+1}\bar{g}_{ij}\| \leq (n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \quad (2.19)$$

At any point $x \in M_t$, we choose an orthonormal basis $\{e_1, \dots, e_n\}$ such that $g_{ij} = \delta_{ij}$, $h_{ij} = \kappa_i \delta_{ij}$, then

$$P_{ij}h_{ij} = \sum_{i,p} 2(\kappa_i \kappa_p - \kappa_i^2) \bar{R}_{pipi} = - \sum_{i < p} 2(\kappa_i - \kappa_p)^2 \bar{R}_{pipi},$$

and

$$-2n\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \leq P_{ij}h_{ij} - \frac{2\bar{r}}{n+1} (\|A\|^2 - \frac{H^2}{n}) \leq 2n\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \quad (2.20)$$

By

$$- \|(\bar{R}_{ij} - \frac{\bar{r}}{n+1} \bar{g}_{ij}) h_{ik} h_{jk}\| \leq \bar{R}_{ij} h_{ik} h_{jk} - \frac{\bar{r}}{n+1} \bar{g}_{ij} h_{ik} h_{jk} \leq \|(\bar{R}_{ij} - \frac{\bar{r}}{n+1} \bar{g}_{ij}) h_{ik} h_{jk}\|$$

we get

$$-(n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \leq \bar{R}_{ij} h_{ik} h_{jk} - \frac{\bar{r}}{n+1} \|A\|^2 \leq (n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \quad (2.21)$$

Now we have

$$\begin{aligned} v &\leq \frac{-4\bar{r}}{n+1} (\|A\|^2 - \frac{H^2}{n}) + 4n\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 + \frac{4\bar{r}}{n+1} \|A\|^2 + 4(n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \\ &\quad - \frac{2\bar{r}}{n+1} \|A\|^2 + 2n\bar{C}\varepsilon_0 e^{-\lambda t} \|A\| \\ &\leq \frac{-2\bar{r}}{n+1} \|A\|^2 + \frac{4\bar{r}}{n(n+1)} H^2 + (8n+4)\bar{C}\varepsilon_0 \|A\|^2 + 2n\bar{C}\varepsilon_0 \|A\|^2 \\ &\leq -2n(1-\varepsilon_0) \|A\|^2 + 4(1+\varepsilon_0) H^2 + (8n+4)\bar{C}\varepsilon_0 \|A\|^2 \\ &\leq -2n \|A\|^2 + 4H^2 + (2n+8n\bar{C}+4\bar{C})\varepsilon_0 \|A\|^2 + 2n\bar{C}\varepsilon_0 \|A\| \\ &\leq -2n \|A\|^2 + 4H^2 + 10n\varepsilon_1 \|A\|^2 + 2n\varepsilon_1 \|A\| \end{aligned} \quad (2.22)$$

where we have used (2.16) in the third inequality and (2.17) in the last inequality. Similarly, using

$$-2\|H(\bar{R}_{ij} - \frac{\bar{r}}{n+1} \bar{g}_{ij}) h_{ij}\| \leq 2H(\bar{R}_{ij} - \frac{\bar{r}}{n+1} \bar{g}_{ij}) h_{ij} \leq 2\|H(\bar{R}_{ij} - \frac{\bar{r}}{n+1} \bar{g}_{ij}) h_{ij}\|$$

we have

$$-2n(n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \leq 2H\bar{R}_{ij} h_{ij} - \frac{2\bar{r}H^2}{n+1} \leq 2n(n+1)\bar{C}\varepsilon_0 e^{-\lambda t} \|A\|^2 \quad (2.23)$$

Now it follows

$$\begin{aligned} uH &\geq \frac{\bar{r}}{n+1} H^2 - 2n(n+1)\bar{C}\varepsilon_0 \|A\|^2 - n^2\bar{C}\varepsilon_0 \|A\|^2 \\ &\geq nH^2 - n\varepsilon_0 H^2 - 2n(n+1)\bar{C}\varepsilon_0 \|A\|^2 - n^2\bar{C}\varepsilon_0 \|A\| \\ &\geq nH^2 - n^2\varepsilon_0 \|A\|^2 - 2n(n+1)\bar{C}\varepsilon_0 \|A\|^2 - n^2\bar{C}\varepsilon_0 \|A\| \\ &\geq nH^2 - 3n^2\varepsilon_1 \|A\|^2 - n^2\varepsilon_1 \|A\| \end{aligned} \quad (2.24)$$

Lemma 2.6 *Inequality (1.4) is preserved under equation (1.2) for all times $0 \leq t < T$, where T is the maximal existence time of the solution to equation (1.2).*

Proof. From Lemma 2.5, we get

$$\begin{aligned} \frac{\partial}{\partial t}(\|A\|^2 - \alpha_n H^2 - 1) &= \Delta(\|A\|^2 - \alpha_n H^2 - 1) - 2(\|\nabla A\|^2 - \alpha_n \|\nabla H\|^2) \\ &\quad + 2\|A\|^2(\|A\|^2 - \alpha_n H^2) + v - 2\alpha_n uH \end{aligned} \quad (2.25)$$

Combining (2.22) and (2.24) gives

$$v - 2\alpha_n uH \leq (22n\varepsilon_1 - 2n)\|A\|^2 + (4 - 2n\alpha_n)H^2 + 6n\varepsilon_1\|A\| \quad (2.26)$$

By taking $\eta = \frac{1}{2^5}$ for $n = 2$ and $\eta = \frac{1}{8(n+2)}$ for $n \geq 3$ in Lemma 2.4, we have

$$\|\nabla A\|^2 \geq \alpha_n \|\nabla H\|^2 - 2^5 n^2 \varepsilon_1^2 \quad (2.27)$$

By substituting (2.26), (2.27) into (2.25), we get

$$\begin{aligned} \frac{\partial}{\partial t}(\|A\|^2 - \alpha_n H^2 - 1) &\leq \Delta(\|A\|^2 - \alpha_n H^2 - 1) + 2\|A\|^2(\|A\|^2 - \alpha_n H^2 - 1) \\ &\quad + (2 + 22n\varepsilon_1 - 2n)\|A\|^2 + (4 - 2n\alpha_n H^2) + 6n\varepsilon_1\|A\| + 2^6 n^2 \varepsilon_1^2 \end{aligned}$$

By $6n\varepsilon_1\|A\| \leq 2n\varepsilon_1\|A\|^2 + 18n\varepsilon_1^2$ and the definition of α_n , a direct computation shows

$$\begin{aligned} &(2 + 22n\varepsilon_1 - 2n)\|A\|^2 + (4 - 2n\alpha_n H^2) + 6n\varepsilon_1\|A\| + 2^6 n^2 \varepsilon_1^2 \\ &< -2(n - 1 - 12\varepsilon_1)(\|A\|^2 - \alpha_n H^2 - 1) \end{aligned}$$

where we have used $\varepsilon_1 \leq \frac{1}{2^7 n}$. Hence

$$\frac{\partial}{\partial t}(\|A\|^2 - \alpha_n H^2 - 1) < \Delta(\|A\|^2 - \alpha_n H^2 - 1) + 2(\|A\|^2 + 1 + 12\varepsilon_1 - n)(\|A\|^2 - \alpha_n H^2 - 1)$$

By the maximum principle, we get the desired inequality. ■

3 A Pinching estimate

In this section we want to show how the eigenvalues of the second fundamental form close to each other when the time becomes large or the mean curvature blows up.

Theorem 3.1 *There exist constants C_1, σ and δ_1 depending on M_0 and n such that it always holds*

$$\|A\|^2 - \frac{H^2}{n} \leq C_1(H^2 + 1)^{1-\sigma} e^{-\delta_1 t},$$

where $\sigma \in (0, 1), t \in [0, T)$.

Proof. For convenience, let

$$W = aH^2 + 1, \quad f_\sigma = \frac{\|A\|^2 - \frac{H^2}{n}}{W^{1-\sigma}},$$

where $a = \alpha_n - \frac{1}{n}$. We use C to denote the constant only depending on n which may vary from line to line. By Lemma 2.6, $f_0 \leq 1$. From Lemma 2.5, we can get the evolution equation of f_0 ,

$$\begin{aligned}\frac{\partial}{\partial t} f_0 &= \frac{1}{W} \frac{\partial}{\partial t} (\|A\|^2 - \frac{H^2}{n}) + (\|A\|^2 - \frac{H^2}{n}) \frac{\partial}{\partial t} (\frac{1}{W}) \\ &= \frac{1}{W} \left\{ \Delta (\|A\|^2 - \frac{H^2}{n}) + (\frac{2}{n} \|\nabla H\|^2 - 2 \|\nabla A\|^2) + 2 (\|A\|^4 - \frac{H^2}{n} \|A\|^2) + v - \frac{2uH}{n} \right\} \\ &\quad - (\|A\|^2 - \frac{H^2}{n}) \frac{2aH}{W^2} (\Delta H + \|A\|^2 H + u)\end{aligned}$$

Using

$$\Delta f_0 = \frac{1}{W} \left\{ \Delta (\|A\|^2 - \frac{H^2}{n}) - f_0 \Delta (aH^2) \right\} - \frac{4aH}{W} \nabla_i H \nabla_i f_0, \quad (3.1)$$

we find

$$\begin{aligned}\frac{\partial}{\partial t} f_0 &= \Delta f_0 + \frac{4aH}{W} \nabla_i H \nabla_i f_0 + \frac{2}{W} (af_0 \|\nabla H\|^2 + \frac{\|\nabla H\|^2}{n} - \|\nabla A\|^2) \\ &\quad + 2 \|A\|^2 f_0 + \frac{1}{W} (v - \frac{2}{n} uH) - \frac{2aH f_0}{W} (\|A\|^2 H + u)\end{aligned} \quad (3.2)$$

By taking η small enough in Lemma 2.4, we have

$$(af_0 + \frac{1}{n}) \|\nabla H\|^2 - \|\nabla A\|^2 \leq \alpha_n \|\nabla H\|^2 - \|\nabla A\|^2 \leq -\frac{\|\nabla H\|^2}{2^4 n} + Ce^{-\lambda t} \quad (3.3)$$

Using (2.20), (2.21) and (2.23), we get

$$\begin{aligned}v &\leq \frac{-4\bar{r}}{n+1} (\|A\|^2 - \frac{H^2}{n}) + \frac{2\bar{r}\|A\|^2}{n+1} + Ce^{-\lambda t} (\|A\|^2 + 1) \\ &\leq \frac{4\bar{r}H^2}{n(n+1)} - \frac{2\bar{r}\|A\|^2}{n+1} + Ce^{-\lambda t} (\|A\|^2 + 1)\end{aligned}$$

and

$$\frac{\bar{r}H^2}{n+1} - Ce^{-\lambda t} (\|A\|^2 + 1) \leq uH \leq \frac{\bar{r}H^2}{n+1} + Ce^{-\lambda t} (\|A\|^2 + 1) \quad (3.4)$$

Now it follows

$$v - \frac{2}{n} Hu \leq -\frac{2\bar{r}}{n+1} (\|A\|^2 - \frac{H^2}{n}) + Ce^{-\lambda t} (\|A\|^2 + 1), \quad (3.5)$$

so we can derive

$$\begin{aligned}
& 2\|A\|^2 f_0 + \frac{1}{W}(v - \frac{2}{n}uH) - \frac{2aHf_0}{W}(\|A\|^2 H + u) \\
& \leq 2f_0\|A\|^2 + \frac{1}{W}\left\{\frac{-2\bar{r}}{n+1}(\|A\|^2 - \frac{H^2}{n}) + C(\|A\|^2 + 1)e^{-\lambda t}\right\} \\
& \quad - \frac{2af_0}{W}\left\{\|A\|^2 H^2 + \frac{\bar{r}H^2}{n+1} - Ce^{-\lambda t}(\|A\|^2 + 1)\right\} \\
& \leq 2f_0\left\{\|A\|^2 - \frac{2\bar{r}}{n+1} - \frac{a\|A\|^2 H^2}{W} - \frac{a\bar{r}H^2}{(n+1)W}\right\} + Ce^{-\lambda t} \\
& \leq \frac{2f_0}{W}\left\{a\|A\|^2 H^2 + \|A\|^2 - \frac{\bar{r}}{n+1}(aH^2 + 1) - aH^2\|A\|^2 - \frac{a\bar{r}H^2}{n+1}\right\} + Ce^{-\lambda t} \\
& \leq \frac{2f_0}{W}\left\{\alpha_n H^2 + 1 - n(1 - \varepsilon_0)(aH^2 + 1) - an(1 - \varepsilon_0)H^2\right\} + Ce^{-\lambda t} \\
& \leq \frac{2f_0}{W}\left\{[(\alpha_n - 2an(1 - \varepsilon_0))]H^2 - n(1 - a)(1 - \varepsilon_0)\right\} + Ce^{-\lambda t} \\
& \leq -\frac{f_0}{2} + Ce^{-\lambda t}
\end{aligned} \tag{3.6}$$

where we have used Lemma 2.6 and (2.16). Substituting (3.3) and (3.6) into (3.2), we have

$$\frac{\partial}{\partial t} f_0 \leq \Delta f_0 + \frac{4aH}{W} \nabla_i H \nabla_i f_0 - \frac{\|\nabla H\|^2}{8nW} - \frac{1}{2} f_0 + Ce^{-\lambda t} \tag{3.7}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial}{\partial t} W^\sigma &= \Delta W^\sigma - 4\sigma(\sigma - 1)a^2 H^2 W^{\sigma-2} \|\nabla H\|^2 - 2a\sigma W^{\sigma-1} \|\nabla H\|^2 \\
&\quad + 2a\sigma H^2 W^{\sigma-1} \|A\|^2 + 2a\sigma u H W^{\sigma-1}
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$\begin{aligned}
\frac{\partial}{\partial t} f_\sigma &= \frac{\partial}{\partial t} (f_0 W^\sigma) \leq \Delta f_\sigma - 2\nabla_i f_0 \nabla_i W^\sigma + 4aH W^{\sigma-1} \nabla_i f_0 \nabla_i H \\
&\quad - \frac{1}{8n} W^{\sigma-1} \|\nabla H\|^2 - \frac{1}{2} f_\sigma + 4a^2 \sigma(1 - \sigma) H^2 W^{\sigma-2} f_0 \|\nabla H\|^2 \\
&\quad - 2a\sigma W^{\sigma-1} f_0 \|\nabla H\|^2 + 2a\sigma H^2 W^{\sigma-1} f_0 (\|A\|^2 + n) + Ce^{-\lambda t} W^\sigma
\end{aligned} \tag{3.9}$$

Using

$$\begin{aligned}
\nabla_i f_0 \nabla_i W^\sigma &= 2a\sigma H W^{\sigma-1} \nabla_i f_0 \nabla_i H \\
\nabla_i f_\sigma \nabla_i H &= W^\sigma \nabla_i f_0 \nabla_i H + 2a\sigma W^{\sigma-1} H f_0 \|\nabla H\|^2
\end{aligned} \tag{3.10}$$

we find

$$\begin{aligned}
& -2\nabla_i f_0 \nabla_i W^\sigma + 4aH W^{\sigma-1} \nabla_i f_0 \nabla_i H \\
& = 4a(1 - \sigma) H W^{\sigma-1} \nabla_i f_\sigma \nabla_i H - 8a^2 \sigma(1 - \sigma) W^{\sigma-2} H^2 f_0 \|\nabla H\|^2
\end{aligned} \tag{3.11}$$

Substitute (3.11) into (3.9), we obtain

$$\begin{aligned} \frac{\partial}{\partial t} f_\sigma &\leq \Delta f_\sigma - 4a(1-\sigma)W^{-1}H\nabla_i f_\sigma \nabla_i H - \frac{1}{8n}W^{\sigma-1}\|\nabla H\|^2 \\ &\quad - \frac{1}{2}f_\sigma + 2\sigma f_\sigma(\|A\|^2 + n) + Ce^{-\lambda t}W^\sigma \end{aligned} \quad (3.12)$$

We can't get the desired estimate by using the maximum principle directly due to the appearance of $2\sigma\|A\|^2 f_\sigma$ on the right hand of (3.12). To proceed further, we may employ the De Giorgi-Moser iteration, see a similar argument in [9]. First, we will show the sectional curvature of M_t is positive. For any point $x \in M_t$, we choose an orthonormal basis $\{e_1, \dots, e_n\}$ for $T_x M_t$, such that $h_{ij} = \kappa_i \delta_{ij}$.

Lemma 3.1 *Let $K_x(e_i, e_j)$ denote the sectional curvature of 2-plane $\text{span}\{e_i, e_j\} \subset T_x(M_t)$. Then*

$$K_x(e_i, e_j) \geq \frac{H^2 + 1}{8n^2}$$

as long as (1.4) holds.

Proof of Lemma 3.1. For any $i \neq j$,

$$\|A\|^2 - \frac{H^2}{n-1} \geq -2\kappa_i \kappa_j$$

By Gauss equation,

$$\begin{aligned} K_x(e_i, e_j) &= \frac{1}{2}(2\bar{R}_{ijij} + 2\kappa_i \kappa_j) \\ &\geq \frac{1}{2}(2 - 4\varepsilon_1 + \frac{H^2}{n-1} - \|A\|^2) \\ &\geq \frac{1}{2}(2 - 4\varepsilon_1 + \frac{H^2}{n-1} - \alpha_n H^2 - 1) \\ &\geq \frac{H^2 + 1}{8n^2} \end{aligned}$$

Since $i \neq j$ is arbitrary, we get the desired estimate. ■

Recall Simon's identity [16],

$$\begin{aligned} \Delta\|A\|^2 &= 2h_{ij}\nabla_i\nabla_j H + 2\|\nabla A\|^2 + 2Z + 2Hh_{ij}\bar{R}_{0i0j} - 2\bar{R}_{00}\|A\|^2 \\ &\quad + 4\bar{R}_{kikp}h_{pj}h_{ij} - 4\bar{R}_{kipj}h_{kp}h_{ij} + 2\bar{\nabla}_k\bar{R}_{0ijk}h_{ij} + 2\bar{\nabla}_i\bar{R}_{0j}h_{ij}, \end{aligned}$$

where $Z = H\text{tr}(A^3) - \|A\|^4$. By a direct computation,

$$\begin{aligned} &2Z + 2Hh_{ij}\bar{R}_{0i0j} - 2\bar{R}_{00}\|A\|^2 + 4\bar{R}_{kikp}h_{pj}h_{ij} - 4\bar{R}_{kipj}h_{kp}h_{ij} + 2\bar{\nabla}_k\bar{R}_{0ijk}h_{ij} + 2\bar{\nabla}_i\bar{R}_{0j}h_{ij} \\ &\geq 2\left(\sum_{i=1}^n \kappa_i\right)\left(\sum_{i=1}^n \kappa_i^3\right) - 2\left(\sum_{i=1}^n \kappa_i^2\right)^2 + \frac{2\bar{r}(n\|A\|^2 - H^2)}{n(n+1)} - Ce^{-\lambda t}(\|A\|^2 + 1) \\ &\geq 2\sum_{i<j} \kappa_i \kappa_j (\kappa_i - \kappa_j)^2 + \frac{2\bar{r}}{n(n+1)} \sum_{i<j} (\kappa_i - \kappa_j)^2 - Ce^{-\lambda t}(\|A\|^2 + 1) \\ &\geq 2\sum_{i<j} K_x(e_i, e_j)(\kappa_i - \kappa_j)^2 - Ce^{-\lambda t}(\|A\|^2 + 1) \\ &\geq \frac{W}{4n}(\|A\|^2 - \frac{H^2}{n}) - Ce^{-\lambda t}(\|A\|^2 + 1) \end{aligned}$$

Now we have

$$\Delta \|A\|^2 \geq 2h_{ij}\nabla_i\nabla_j H + 2\|\nabla A\|^2 + \frac{W}{4n}(\|A\|^2 - \frac{H^2}{n}) - Ce^{-\lambda t}(\|A\|^2 + 1) \quad (3.13)$$

Substituting the inequality above into (3.1) gives

$$\begin{aligned} \Delta f_0 \geq & W^{-1}\{2h_{ij}\nabla_i\nabla_j H + 2\|\nabla A\|^2 + \frac{W}{4n}(\|A\|^2 - \frac{H^2}{n}) - Ce^{-\lambda t}(\|A\|^2 + 1) \\ & - \frac{2}{n}H\Delta H - \frac{2}{n}\|\nabla H\|^2 - 2af_0H\Delta H - 2af_0\|\nabla H\|^2\} - 4aHW^{-1}\nabla_i H\nabla_i f_0 \end{aligned}$$

We denote by $h_{ij}^0 = h_{ij} - \frac{1}{n}Hg_{ij}$ the tracefree second fundamental form. Notice

$$\frac{2}{n}\|\nabla H\|^2 + 2af_0\|\nabla H\|^2 \leq 2\alpha_n\|\nabla H\|^2 \leq 2\|\nabla A\|^2 + Ce^{-\lambda t}$$

Then we derive

$$\begin{aligned} \Delta f_0 \geq & W^{-1}\{2h_{ij}^0\nabla_i\nabla_j H + \frac{W}{4n}(\|A\|^2 - \frac{H^2}{n}) - Ce^{-\lambda t}(\|A\|^2 + 1) \\ & - 2af_0H\Delta H - 4aH\nabla_i H\nabla_i f_0\} \end{aligned} \quad (3.14)$$

Multiplying two sides of (3.14) by W^σ yields

$$\begin{aligned} \Delta f_\sigma = & W^\sigma \Delta f_0 + f_0 \Delta W_\sigma + 2\nabla_i f_0 \nabla_i W^\sigma \\ \geq & W^{\sigma-1}\{2h_{ij}^0\nabla_i\nabla_j H + \frac{1}{4n}W^{2-\sigma}f_\sigma - 2af_0H\Delta H - 4aH\nabla_i H\nabla_i f_0\} \\ & + f_0\{\sigma W^{\sigma-1}(2aH\Delta H + 2a\|\nabla H\|^2) + 4a^2\sigma(\sigma-1)H^2W^{\sigma-2}\|\nabla H\|^2\} \\ & + 2\nabla_i f_0 \nabla_i W^\sigma - Ce^{-\lambda t}(\|A\|^2 + 1)W^{\sigma-1} \end{aligned}$$

From (3.11), we have

$$\begin{aligned} & -4aHW^{\sigma-1}\nabla_i H\nabla_i f_0 + 4a^2\sigma(\sigma-1)f_0W^{\sigma-1}\|\nabla H\|^2 + 2\nabla_i f_0 \nabla_i W^\sigma \\ = & -4a(1-\sigma)HW^{-1}\nabla_i H\nabla_i f_\sigma + 4a^2\sigma(1-\sigma)f_0H^2W^{\sigma-2}\|\nabla H\|^2 \end{aligned}$$

Since $\sigma < 1$, we get

$$\begin{aligned} \Delta f_\sigma \geq & 2W^{\sigma-1}h_{ij}^0\nabla_i\nabla_j H + \frac{W}{4n}f_\sigma - 2a(1-\sigma)HW^{-1}f_\sigma\Delta H \\ & - 4a(1-\sigma)HW^{-1}\nabla_i H\nabla_i f_\sigma - Ce^{-\lambda t}(\|A\|^2 + 1)W^{\sigma-1} \end{aligned}$$

By multiplying this inequality by f_σ^{p-1} and integrating on M_t , it follows

$$\begin{aligned} \frac{1}{4n} \int W f_\sigma^p d\mu \leq & -(p-1) \int f_\sigma^{p-2} \|\nabla f_\sigma\|^2 d\mu - 2 \int W^{\sigma-1} h_{ij}^0 \nabla_i \nabla_j H f_\sigma^{p-1} d\mu \\ & + 2a(1-\sigma) \int HW^{-1} f_\sigma^p \Delta H d\mu + 4a(1-\sigma) \int HW^{-1} f_\sigma^{p-1} \nabla_i H \nabla_i f_\sigma d\mu \\ & + Ce^{-\lambda t} \int (\|A\|^2 + 1) W^{\sigma-1} f_\sigma^{p-1} d\mu \end{aligned} \quad (3.15)$$

By Codazzi equation,

$$\nabla_i h_{ij}^0 = \frac{n-1}{n} \nabla_j H + \bar{R}_{0j}$$

Using Stokes' theorem, then

$$\begin{aligned} & 2 \int W^{\sigma-1} h_{ij}^0 \nabla_i \nabla_j H f_\sigma^{p-1} d\mu \\ & \geq -2 \int W^{\sigma-1} \{ (p-1) \|\nabla H\| \|h_{ij}^0\| f_\sigma^{p-2} + \frac{n-1}{n} \|\nabla H\|^2 f_\sigma^{p-1} + e^{-\lambda t} \|\nabla H\| f_\sigma^{p-1} \} d\mu \\ & \quad - \int 4a(1-\sigma) W^{\sigma-2} \|h_{ij}^0\| \|H\| \|\nabla H\|^2 f_\sigma^{p-1} d\mu \end{aligned} \quad (3.16)$$

and

$$\begin{aligned} & 2a(1-\sigma) \int H W^{-1} f_\sigma^p \Delta H d\mu \\ & \geq -2a \int \|\nabla H\|^2 W^{-1} f_\sigma^p \\ & \quad - 2a \int (2aH^2 W^{-2} f_\sigma^p \|\nabla H\|^2 + p \|H\| W^{-1} f_\sigma^{p-1} \|\nabla H\| \|\nabla f_\sigma\|) d\mu \end{aligned} \quad (3.17)$$

Combining (3.15), (3.16) and (3.17), we obtain

$$\begin{aligned} & \frac{1}{4n} \int W f_\sigma^p d\mu \\ & \leq -(p-1) \int f_\sigma^{p-2} \|\nabla f_\sigma\|^2 d\mu + 2(p-1) \int W^{\sigma-1} \|\nabla H\| \|h_{ij}^0\| \|\nabla f_\sigma\| f_\sigma^{p-2} d\mu \\ & \quad + \int \|\nabla H\|^2 (W^{\sigma-1} f_\sigma^{p-1} + 2a W^{-1} f_\sigma^p) d\mu + 4 \int W^{\sigma-1} e^{-\lambda t} \|\nabla H\| f_\sigma^{p-1} d\mu \\ & \quad + \int 4a(1-\sigma) W^{\sigma-2} \|h_{ij}^0\| \|H\| \|\nabla H\|^2 f_\sigma^{p-1} d\mu + 4a^2 \int H^2 W^{-2} f_\sigma^p \|\nabla H\|^2 d\mu \\ & \quad + 2a(2+p) \int \|H\| W^{-1} f_\sigma^{p-1} \|\nabla H\| \|\nabla f_\sigma\| d\mu + C e^{-\lambda t} \int (\|A\|^2 + 1) W^{\sigma-1} f_\sigma^{p-1} d\mu \end{aligned}$$

Using

$$\|h_{ij}^0\|^2 = \|A\|^2 - \frac{H^2}{n} = f_\sigma W^{1-\sigma}, \quad \|aH\| \leq W^{\frac{1}{2}}, \quad f_\sigma \leq W^\sigma$$

and Cauchy-Schwartz inequality, we derive

Lemma 3.2 *Let $p \geq 2$. Then for any $\theta > 0$ and any $\sigma \in [0, \frac{1}{4}]$, it holds*

$$\begin{aligned} \frac{1}{4n} \int W f_\sigma^p d\mu & \leq (2\theta(p+1) + 8) \int W^{\sigma-1} f_\sigma^{p-1} \|\nabla H\|^2 d\mu \\ & \quad + \frac{2p}{\theta} \int f_\sigma^{p-2} \|\nabla f_\sigma\|^2 d\mu + C e^{-\lambda t} \int W^\sigma f_\sigma^{p-1} d\mu \end{aligned}$$

Now we are ready to give an estimate for L^p -norm of f_σ , if σ is of order $p^{-\frac{1}{2}}$.

Lemma 3.3 For any $p \geq 2^6 n^2, \sigma \leq 2^{-6} n^{-2} p^{-\frac{1}{2}}$, there exist constants C^* and $\delta > 0$ depending only on M_0 and n , such that for any $t \in [0, T)$ we have the estimate

$$\left(\int_{M_t} f_\sigma^p d\mu \right)^{\frac{1}{p}} \leq C^* e^{-\delta t}.$$

Proof of Lemma 3.3 From (3.12), it's easy to show

$$\begin{aligned} & \frac{\partial}{\partial t} \int f_\sigma^p d\mu + p(p-1) \int f_\sigma^{p-2} \|\nabla f_\sigma\|^2 d\mu + \frac{p}{8} \int W^{\sigma-1} f_\sigma^{p-1} \|\nabla H\|^2 d\mu \\ & \leq 4ap \int \|H\| W^{-1} \|\nabla H\| \|\nabla f_\sigma\| f_\sigma^{p-1} d\mu + 2\sigma p \int (\|A\|^2 + n) f_\sigma^p d\mu - \frac{p}{2} \int f_\sigma^p d\mu \\ & \quad + Cpe^{-\lambda t} \int W^\sigma f_\sigma^{p-1} d\mu + \int \|g^{ij}(\bar{R}_{ij} - \frac{\bar{r}\bar{g}_{ij}}{n+1})\| f_\sigma^p d\mu - \int H^2 f_\sigma^p d\mu \\ & \leq 4ap \int \|H\| W^{-1} \|\nabla H\| \|\nabla f_\sigma\| f_\sigma^{p-1} d\mu + (8n\sigma p + 1) \int W f_\sigma^p d\mu - \frac{p}{2} \int f_\sigma^p d\mu \\ & \quad + Cpe^{-\lambda t} \int W^\sigma f_\sigma^{p-1} d\mu - \int W f_\sigma^p d\mu, \end{aligned}$$

where we have used (2.19) and the fact that

$$\|A\|^2 + n \leq \alpha_n H^2 + 1 + n \leq 4nW, \quad f_\sigma \leq W^\sigma$$

Set $\theta = \frac{1}{16\sqrt{p}}$ in Lemma 3.4, then by our choice of p and σ ,

$$\frac{\partial}{\partial t} \int f_\sigma^p d\mu \leq -\frac{p}{2} \int f_\sigma^p d\mu + Cpe^{-\lambda t} \int W^\sigma f_\sigma^{p-1} d\mu - \int W f_\sigma^p d\mu \quad (3.18)$$

Since

$$\begin{aligned} \frac{d}{dt} \int_{M_t} d\mu &= \int_{M_t} (-H^2 - g^{ij} \bar{R}_{ij} + \frac{n\bar{r}}{n+1}) d\mu \\ &\leq \int_{M_t} \|g^{ij}(\bar{R}_{ij} - \frac{\bar{r}\bar{g}_{ij}}{n+1})\| d\mu \leq ne^{-\lambda t} \int_{M_t} d\mu \end{aligned}$$

then

$$\int_{M_t} d\mu \leq e^{\frac{n}{\lambda}} \int_{M_0} d\mu.$$

Let

$$\Lambda = 1 + e^{\frac{n}{\lambda}} \int_{M_0} d\mu,$$

and

$$I = \{t \in [0, T) \mid \int_{M_t} f_\sigma^p > \Lambda e^{-\frac{p\lambda t}{2}}\}.$$

If $I = \emptyset$, then the Lemma follows automatically. Otherwise, let $t_0 = \inf I$, then at $t = t_0$, we have

$$\int_{M_{t_0}} f_\sigma^p = \Lambda e^{-\frac{p\lambda t_0}{2}}.$$

For any $t_1 \in (t_0, T)$, we only need to consider the case

$$\int_{M_t} f_\sigma^p \geq \Lambda e^{-\frac{p\lambda t}{2}}, \quad \forall t \in [t_0, t_1]$$

Let $s \in (1, \frac{3}{2})$ satisfy $\sigma + \frac{1}{s} = 1$. Then by Hölder inequality and Young's inequality,

$$\begin{aligned} \int_{M_t} W^\sigma f_\sigma^{p-1} &= \int_{M_t} W^\sigma f_\sigma^{p\sigma} f_\sigma^{p-p\sigma-1} \\ &\leq \left(\int_{M_t} W f_\sigma^p \right)^\sigma \left(\int_{M_t} f_\sigma^{(p-p\sigma-1)s} \right)^{\frac{1}{s}} \\ &\leq \tau \int_{M_t} W f_\sigma^p + \left(\frac{1}{\tau} \right)^{s\sigma} \int_{M_t} f_\sigma^{p-s} \\ &\leq \tau \int_{M_t} W f_\sigma^p + \Lambda \left(\frac{1}{\tau} \right)^{s\sigma} \left(\int_{M_t} f_\sigma^p \right)^{1-\frac{s}{p}} \end{aligned} \quad (3.19)$$

By choosing $\tau = \frac{1}{Cp}$ and substituting (3.19) into (3.18),

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p \leq -\frac{p}{2} \int_{M_t} f_\sigma^p d\mu + 2\Lambda C p e^{-\lambda t} \left(\int_{M_t} f_\sigma^p d\mu \right)^{1-\frac{s}{p}}$$

Since $\int_{M_t} f_\sigma^p d\mu \geq \Lambda e^{-\frac{p\lambda t}{2}}$, we have

$$\frac{\partial}{\partial t} \int_{M_t} f_\sigma^p d\mu \leq -\frac{p}{2} \int_{M_t} f_\sigma^p d\mu + 2\Lambda C p e^{-\frac{\lambda t}{4}} \int_{M_t} f_\sigma^p$$

Integrating the inequality above from t_0 to t_1 yields

$$\int_{M_{t_1}} f_\sigma^p \leq e^{\frac{8\Lambda C}{\lambda} p} \int_{M_{t_0}} f_\sigma^p e^{-\frac{p}{2}(t_1-t_0)} \leq (C^*)^p e^{-p\delta t_1}$$

where $C^* = \Lambda e^{\frac{8\Lambda C}{\lambda}}$, $\delta = \min\{\frac{\lambda}{2}, \frac{1}{2}\}$. Note the constant C^* is independent of t_1 , thus we complete the proof of the Lemma.

As a consequence of Lemma 3.3, we have

Corollary 3.1 *For any $m \geq 1$, $p \geq m^2 2^7 n^2$, and $\sigma \leq 2^{-7} n^{-2} p^{-\frac{1}{2}}$, we have*

$$\left(\int_{M_t} W^m f_\sigma^p \right)^{\frac{1}{p}} \leq C^* e^{-\delta t}$$

To prove Theorem 3.1, it suffices to give an uniformly upper bound for $g_\sigma = f_\sigma e^{\frac{\delta t}{2}}$. For any m, p, σ satisfying the condition of Corollary 3.1,

$$\left(\int_{M_t} W^m g_\sigma^p \right)^{\frac{1}{p}} \leq e^{\frac{\delta t}{2}} \left(\int_{M_t} W^m f_\sigma^p \right)^{\frac{1}{p}} \leq C^* e^{-\frac{\delta t}{2}}. \quad (3.20)$$

Let $g_{\sigma,k} = \max(g_\sigma - k, 0)$, $\varphi = g_{\sigma,k}^{\frac{p}{2}}$, $A(k, t) = \{x \in M_t \mid g_\sigma > k\}$, and

$$\| \| A(k, t) \| \|_{T_1} = \int_0^{T_1} \| A(k, t) \| dt = \int_0^{T_1} \int_{A(k,t)} d\mu dt$$

By Hölder inequality,

$$\|A(k, t)\| \leq \frac{1}{k} \int_{M_t} g_\sigma d\mu \leq \frac{\Lambda}{k} e^{\frac{\delta t}{2}} \left(\int_{M_t} f_\sigma^p \right)^{\frac{1}{p}} d\mu \leq \frac{\Lambda C^*}{k}$$

From (3.20), we derive

$$\int_{A(k, t)} \|H\|^n d\mu \leq \left(\frac{1}{a}\right)^{\frac{n}{2}} k^{-p} \int_{M_t} W^{\frac{n}{2}} g_\sigma^p d\mu \leq (2n)^n \left(\frac{C^*}{k}\right)^p \quad (3.21)$$

Given $p \geq 2$, we can choose $k_1 \geq 2nCC^*$ large enough such that for any $k \geq k_1$ the following Sobolev inequality [6] holds

$$\left(\int_{A(k, t)} \varphi^{\frac{n}{n-1}} d\mu \right)^{\frac{n-1}{n}} \leq c_n \left(\int_{A(k, t)} \|\nabla \varphi\|^2 d\mu + \int_{A(k, t)} \|H\| \varphi d\mu \right)$$

where c_n is a constant only depending on n . By Hölder inequality,

$$\left(\int_{A(k, t)} \varphi^{2q} d\mu \right)^{\frac{1}{q}} \leq c_n \int_{A(k, t)} \|\nabla \varphi\|^2 d\mu + c_n \left(\int_{A(k, t)} \|H\|^n d\mu \right)^{\frac{2}{n}} \left(\int_{A(k, t)} \varphi^{2q} d\mu \right)^{\frac{1}{q}} \quad (3.22)$$

where

$$q = \begin{cases} \frac{n}{n-2}, & n > 2, \\ < \infty, & n = 2. \end{cases}$$

Since k_1 is large enough,

$$c_n \left(\int_{A(k, t)} \|H\|^n d\mu \right)^{\frac{2}{n}} \leq \frac{1}{2}$$

By (3.12), we have

$$\frac{\partial}{\partial t} \int_{A(k, t)} \varphi^2 + \int_{A(k, t)} \|\nabla \varphi\|^2 \leq Cp \int_{A(k, t)} W g_\sigma^p d\mu \quad (3.23)$$

Substituting (3.22) into (3.23) gives

$$\frac{\partial}{\partial t} \int_{A(k, t)} \varphi^2 d\mu + \frac{1}{C} \left(\int_{A(k, t)} \varphi^{2q} d\mu \right)^{\frac{1}{q}} \leq Cp \int_{A(k, t)} W g_\sigma^p d\mu$$

Then for any $T_1 < T$,

$$\sup_{[0, T_1]} \int_{M_t} \varphi^2 d\mu + \frac{1}{C} \int_0^{T_1} \left(\int_{M_t} \varphi^{2q} d\mu \right)^{\frac{1}{q}} dt \leq Cp \int_0^{T_1} \int_{A(k, t)} W g_\sigma^p d\mu dt \quad (3.24)$$

Using interpolation inequalities for L^p -space, we have

$$\left(\int_{M_t} \varphi^{2q_0} d\mu \right)^{\frac{1}{q_0}} \leq \left(\int_{M_t} \varphi^{2q} d\mu \right)^{\frac{\eta}{q}} \left(\int_{M_t} \varphi^2 d\mu \right)^{1-\eta}$$

where $1 < q_0 < q$ and $\eta = \frac{1}{q_0} = \frac{1}{2-\frac{1}{q}}$.

Thus

$$\begin{aligned}
& \left(\int_0^{T_1} \int_{A(k,t)} \varphi^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} \\
& \leq \left[\int_0^{T_1} \left(\int_{A(k,t)} \varphi^{2q} d\mu \right)^{\frac{\eta q_0}{q}} \left(\int_{A(k,t)} \varphi^2 d\mu \right)^{(1-\eta)q_0} dt \right]^{\frac{1}{q_0}} \\
& \leq \left(\sup_{[0,T_1]} \int_{M_t} \varphi^2 d\mu \right)^{1-\frac{1}{q_0}} \left(\int_0^{T_1} \left(\int_{M_t} \varphi^{2q} d\mu \right)^{\frac{1}{q}} dt \right)^{\frac{1}{q_0}} \\
& \leq Cp \int_0^{T_1} \int_{A(k,t)} W g_\sigma^p d\mu dt \\
& \leq Cp \|A(k,t)\|^{1-\frac{1}{\theta}} \left(\int_0^{T_1} \int_{M_t} W^\theta g_\sigma^{p\theta} d\mu dt \right)^{\frac{1}{\theta}}
\end{aligned}$$

where $\theta > 1$ is a positive constant to be chosen, we have used (3.24) in the third inequality. Applying Hölder inequality again, we have

$$\begin{aligned}
\int_0^{T_1} \int_{M_t} \varphi^2 d\mu dt & \leq \|A(k,t)\|^{1-\frac{1}{q_0}} \left(\int_0^{T_1} \int_{A(k,t)} \varphi^{2q_0} d\mu dt \right)^{\frac{1}{q_0}} \\
& \leq Cp \|A(k,t)\|^{2-\frac{1}{q_0}-\frac{1}{\theta}} \left(\int_0^{T_1} \int_{A(k,t)} W^\theta g_\sigma^{p\theta} d\mu dt \right)^{\frac{1}{\theta}}. \tag{3.25}
\end{aligned}$$

Now we choose θ large such that $\gamma = 2 - \frac{1}{q_0} - \frac{1}{\theta} > 1$. Notice that θ is independent of the choice of p . Choosing fixed $p_1 \geq \theta 2^8 n^2$ and $\sigma_1 \leq 2^8 n^{-2} p_1^{-\frac{1}{2}}$, by (3.20), we have

$$\left(\int_0^{T_1} \int_{A(k,t)} W^\theta g_{\sigma_1}^{p_1 \theta} d\mu dt \right)^{\frac{1}{\theta}} \leq \left(\int_0^{T_1} (C e^{-\frac{\delta t}{2}})^{p_1 \theta} dt \right)^{\frac{1}{\theta}} \leq C_{p_1} \tag{3.26}$$

Together (3.25) with (3.26), we obtain

$$\|h - k\|^{p_1} \|A(h,t)\|_{T_1} \leq \int_0^{T_1} \int_{M_t} \varphi^2 d\mu dt \leq C_{p_1} \|A(k)\|_{T_1}^\gamma, \quad h > k > k_1.$$

Thus by the De Giorgi's iteration Lemma, we conclude

$$\|A(k,t)\| = 0, \quad \forall k \geq k_1 + d,$$

where d is a constant depending on M_0, n and λ . Hence

$$g_\sigma \leq k_1 + d.$$

Notice both k_1 and d are independent of T_1 , so we finish the proof. ■

4 The gradient estimate

In this section, we use Theorem 3.1 to get an estimate for the gradient of mean curvature.

Theorem 4.1 For any $0 < \beta \leq 1$, there exists a constant C_β depending only on \bar{g}_0, M_0, n and β , such that for any point $(x, t) \in M_t \times [0, T)$ we have

$$\|\nabla H\|^2 \leq (\beta\|H\|^4 + C_\beta)e^{-\frac{\delta_1}{2}t}$$

Proof. By Lemma 2.3,

$$\begin{aligned} \frac{\partial}{\partial t}\|\nabla H\|^2 &= -\left(\frac{\partial}{\partial t}g_{ij}\right)\nabla_i H \nabla_j H + 2\nabla_i H \nabla_i \left(\frac{\partial H}{\partial t}\right) \\ &= (2Hh_{ij} + 2\bar{R}_{ij} - \frac{\bar{r}}{n+1}g_{ij})\nabla_i H \nabla_j H + 2\nabla_i H \nabla_i (\Delta H + \|A\|^2 H + u) \\ &\leq \Delta\|\nabla H\|^2 - 2\|\nabla^2 H\|^2 + 2\|A\|^2\|\nabla H\|^2 + 2H\nabla_i H \nabla_i \|A\|^2 + 2h_{ij}h_{jp}\nabla_i H \nabla_p H \\ &\quad - 2\bar{R}_{ijpj}\nabla_i H \nabla_p H + C\|\nabla A\|^2 + 2\|\nabla H\|\|\nabla u\| \\ &\leq \Delta\|\nabla H\|^2 + C_2(H^2 + 1)\|\nabla A\|^2 + C_2e^{-\lambda t} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\frac{\partial}{\partial t}\left(H^2(\|A\|^2 - \frac{H^2}{n})\right) \\ &= H^2\frac{\partial}{\partial t}\left(\|A\|^2 - \frac{H^2}{n}\right) + \left(\frac{\partial}{\partial t}H^2\right)\left(\|A\|^2 - \frac{1}{n}H^2\right) \\ &= H^2\left\{\Delta\left(\|A\|^2 - \frac{1}{n}H^2\right) - 2\left(\|\nabla A\|^2 - \frac{\|\nabla H\|^2}{n}\right) + 2\|A\|^2\left(\|A\|^2 - \frac{H^2}{n}\right) + \left(v - \frac{2}{n}uH\right)\right\} \\ &\quad + \left(\Delta H^2 - 2\|\nabla H\|^2 + 2\|A\|^2H^2 + 2uH\right)\left(\|A\|^2 - \frac{H^2}{n}\right) \\ &= \Delta\left(H^2\left(\|A\|^2 - \frac{H^2}{n}\right)\right) - 4H\nabla_i H \nabla_i \left(\|A\|^2 - \frac{H^2}{n}\right) - 2H^2\left(\|\nabla A\|^2 - \frac{\|\nabla H\|^2}{n}\right) \\ &\quad - 2\left(\|A\|^2 - \frac{H^2}{n}\right)\|\nabla H\|^2 + 4\|A\|^2H^2\left(\|A\|^2 - \frac{H^2}{n}\right) + H^2\left(v - \frac{2}{n}uH\right) + 2uH\left(\|A\|^2 - \frac{H^2}{n}\right) \end{aligned} \tag{4.2}$$

By applying Theorem 3.1 we can give an estimate of the second term of (4.2) ,

$$\begin{aligned} \|4H\nabla_i H \nabla_i \left(\|A\|^2 - \frac{H^2}{n}\right)\| &= \|8H\nabla_i H h_{kl}^0 \nabla_i h_{kl}^0\| \\ &\leq 8\|H\|\|\nabla A\|\|\nabla H\|\|h_{kl}^0\| \\ &\leq 8n\|H\|\|\nabla A\|^2 C_1(H^2 + 1)^{\frac{1-\sigma}{2}} \\ &\leq \frac{1}{4n}H^2\|\nabla A\|^2 + C_2\|\nabla A\|^2 \end{aligned} \tag{4.3}$$

By Lemma 2.4, we can choose $\eta > 0$ such that

$$\|\nabla A\|^2 - \frac{1}{n}\|\nabla H\|^2 \geq \frac{1}{4n}\|\nabla A\|^2 - C_2e^{-\lambda t} \tag{4.4}$$

Combining (3.2) and (3.3) gives

$$H^2\left(v - \frac{2}{n}uH\right) + 2uH\left(\|A\|^2 - \frac{H^2}{n}\right) \leq C_2(\|A\|^4 + 1)e^{-\lambda t} \tag{4.5}$$

Substituting (4.3), (4.4) and (4.5) into (4.2) yields

$$\begin{aligned} \frac{\partial}{\partial t}(H^2(\|A\|^2 - \frac{H^2}{n})) &\leq \Delta(H^2(\|A\|^2 - \frac{H^2}{n})) - \frac{1}{4n}H^2\|\nabla A\|^2 + 4\|A\|^2H^2(\|A\|^2 - \frac{H^2}{n}) \\ &\quad + C_2\|\nabla A\|^2 + C_2(\|A\|^4 + 1)e^{-\lambda t} \end{aligned} \quad (4.6)$$

Similarly, we can get the following estimate

$$\begin{aligned} &\frac{\partial}{\partial t}(\|A\|^2 - \frac{H^2}{n}) \\ &= \Delta(\|A\|^2 - \frac{H^2}{n}) - 2(\|\nabla A\|^2 - \frac{\|\nabla H\|^2}{n}) + 2\|A\|^2(\|A\|^2 - \frac{H^2}{n}) + v - \frac{2}{n}uH \\ &\leq \Delta(\|A\|^2 - \frac{H^2}{n}) - \frac{1}{2n}\|\nabla A\|^2 + 2\|A\|^2(\|A\|^2 - \frac{H^2}{n}) + C_2(\|A\|^2 + 1)e^{-\lambda t} \end{aligned} \quad (4.7)$$

Let $\Psi = H^2(\|A\|^2 - \frac{H^2}{n}) + 4nC_2(\|A\|^2 - \frac{H^2}{n})$. Then it follows from (4.6) and (4.7)

$$\begin{aligned} \frac{\partial}{\partial t}\Psi &\leq \Delta\Psi - (\frac{H^2}{4n} + C_2)\|\nabla A\|^2 + 4\|A\|^2(\|A\|^2 - \frac{H^2}{n})(H^2 + 2nC_2) \\ &\quad + 5nC_2^2(\|A\|^4 + 1)e^{-\lambda t} \end{aligned}$$

A direct computation shows

$$\begin{aligned} \frac{\partial}{\partial t}\|A\|^4 &= 2\|A\|^2(\Delta\|A\|^2 - 2\|\nabla A\|^2 + 2\|A\|^4 + v) \\ &\geq \Delta\|A\|^4 - 12\|A\|^2\|\nabla A\|^2 + 4\|A\|^6 - 4n\|A\|^2(\|A\|^2 - \frac{1}{n}H^2) \\ &\quad - C_2(\|A\|^4 + 1)e^{-\lambda t} \end{aligned} \quad (4.8)$$

Now consider the function

$$\Phi = e^{\frac{\delta_1 t}{2}}(\|\nabla H\|^2 + C_3\Psi) - \beta\|A\|^4$$

Choose $C_3 \geq 12nC_2$, then there exists a constant C_4 such that

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + C_4(\|A\|^4 + 1)(\|A\|^2 - \frac{H^2}{n})e^{\frac{\delta_1 t}{2}} + C_4(\|A\|^4 + 1)e^{-\lambda t} - 4\beta\|A\|^6$$

Applying Theorem 3.1 and Young's inequality, we obtain

$$\frac{\partial}{\partial t}\Phi \leq \Delta\Phi + C_5e^{-\frac{\delta_1 t}{2}}$$

Therefore Φ is bounded by a constant C_6 . Hence

$$\|\nabla H\|^2 \leq (\beta\|A\|^4 + C_6)e^{-\frac{\delta_1 t}{2}}.$$

We complete the proof by Lemma 2.6. ■

5 Convergence of the hypersurface

In this section, we use Theorem 3.1 and Theorem 4.1 to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We consider the following two cases.

Case 1: $\max_{M_t} \|H\| \rightarrow \infty$ as $t \rightarrow T$. By Theorem 4.1, we always have

$$\|\nabla H\| \leq \beta^2 H^2 + C_\beta \quad \text{on } t \in [0, T).$$

Let

$$H_{\max}(t) = \max_{M_t} H, \quad H_{\min}(t) = \min_{M_t} H$$

Suppose $\max_{M_t} \|H\| = H_{\max}(t) > 0$. For any $\beta > 0$, there exists some θ depending on β with $C_\beta \leq \beta^2 H_{\max}^2$ at $t = \theta$, so $\|\nabla H\| \leq 2\beta^2 H_{\max}^2$. Let x_0 be the point where H attains its maximum. Then for any point x with $d(x, x_0) \leq \frac{1}{\beta H_{\max}}$, we have

$$H(x) \geq H_{\max} - 2d(x, x_0)\beta^2 H_{\max}^2 \geq (1 - 2\beta)H_{\max}$$

and the sectional curvature $K_{M_t}(x)$ of M_t satisfies

$$K_{M_t}(x) \geq \frac{H^2}{8n^2} \geq \frac{(1 - 2\beta)^2 H_{\max}^2}{8n^2},$$

By Myers' theorem, any geodesic starting from x_0 with length larger than $\frac{2\sqrt{2}n\pi}{1-2\beta} H_{\max}^{-1}$ must have conjugate points. By choosing β small, we can get

$$H_{\min} \geq (1 - 2\beta)H_{\max} \quad \text{on } M_\theta$$

Thus by a suitable choice of θ we know the mean curvature of the hypersurface is positive and can be arbitrarily large. Moreover, at some $t = \theta$ the inequality below holds everywhere on M_θ

$$\|A\|^2 \leq \alpha_n H^2 + 1 < \frac{1}{n-1} H^2$$

Hence M_θ is strictly convex. By the maximum principle, the maximal existence time of the equation (1.2) must be finite. By a similar argument as Huisken in [9], we know M_t converge to a round point.

Case 2: $\|H\|$ is uniformly bounded and $T = \infty$. Now

$$\|A\|^2 - \frac{H^2}{n} \leq C e^{-\delta_1 t}, \quad \|\nabla H\|^2 \leq C e^{-\frac{\delta_1 t}{2}}$$

Furthermore, we can get

Claim : $H_{\max}(t) > -\tilde{C}e^{-\frac{\delta_1 t}{4}}$ and $H_{\min}(t) < \tilde{C}e^{-\frac{\delta_1 t}{4}}$ for some large constant $\tilde{C} > C$.

Suppose there exists a moment t_0 such that $H_{\max}(t) \leq -\tilde{C}e^{-\frac{\delta_1 t}{4}}$ at $t = t_0$. Note $\delta_1 \leq \frac{\lambda}{2}$. From (3) in Lemma 2.5, at $t = t_0$, we have

$$\begin{aligned} \frac{\partial H_{\max}}{\partial t} &\leq \|A\|^2 H_{\max} + H_{\max} + C e^{-\lambda t} \\ &\leq -\tilde{C}e^{-\delta_1 t} + C e^{-\lambda t} \\ &< 0 \end{aligned}$$

It follows

$$H_{\max}(t) \leq -\tilde{C}e^{-\frac{\delta_1 t_0}{4}}, \forall t \in [t_0, \infty)$$

which contradicts with the fact that $H(\cdot, t) \rightarrow 0$ as $t \rightarrow \infty$. The other inequality $H_{\min}(t) < \tilde{C}e^{-\frac{\delta_1 t}{4}}$ can be derived by the same way.

On the other hand, by Lemma 3.2, the Ricci curvature of M_t is no less than $\frac{1}{8n}$, thus the diameter of M_t is smaller than $2\sqrt{2}n\pi$. Since $\|\nabla H\|^2 \leq Ce^{-\frac{\delta_1 t}{2}}$, we have

$$\|H_{\max}(t) - H_{\min}(t)\| \leq Ce^{-\frac{\delta_1 t}{4}}$$

Then it follows $\|H\|^2 \leq 4\tilde{C}^2e^{-\frac{\delta_1 t}{2}}$ and $\|A\| \leq 5\tilde{C}^2e^{-\frac{\delta_1 t}{4}}$. One can show the exponentially decreasing for $\|\nabla^m A\|$ by the similar argument as [9]. Since $(N^{n+1}, \bar{g}(t))$ converge to S^{n+1} in C^∞ -topology, so we get the C^∞ -convergence to the totally geodesic sphere for M_t . ■

References

- [1] J. Cheeger, D. Ebin, Comparison Theorems in Riemannian Geometry, Amsterdam: North-Holland, 1975.
- [2] J. Cheeger, M. Gromov, and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Diff. Geom. 17(1982), 15-53.
- [3] T. Colding and W. Minicozzi, Generic mean curvature flow I; generic singularities, Ann. of Math. 175 (2012), 755-833.
- [4] R. Hamilton, Three-manifolds with positive Ricci curvatures. J. Diff. Geom. 17 (1982), 255-306.
- [5] X. Han and J. Li, A mean curvature flow along a Kähler-Ricci flow, arXiv: 1105.1200v1[math.DG].
- [6] D. Hoffman and J. Spruck, Sobolev and isoperimetric inequalities for Riemannian submanifolds, Comm. Pure Appl. Math. 27 (1974), 715-727.
- [7] G. Huisken, Flow by mean curvature of convex surfaces into spheres, J. Diff. Geom. 20 (1984), 237-266.
- [8] G. Huisken, Ricci deformation of the metric on a Riemannian manifold, J. Diff. Geom. 21 (1985), 47-62.
- [9] G. Huisken, Contracting convex hypersurfaces in Riemannian manifold by their mean curvature, Invent. Math. 84 (1986), 463-480.
- [10] G. Huisken, Deforming hypersurfaces of the sphere by their mean curvature, Math. Z. 195 (1987), 205-219.
- [11] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, J. Diff. Geom. 31 (1990), 285-299.

- [12] J. Lott, Mean curvature flow in a Ricci flow background, *Comm. Math. Phys.* 313 (2012), 517-533.
- [13] A. Magni, C. Mantegazza and E. Tsatis, Flow by mean curvature inside a moving ambient space, *J. Evol. Equ.* 13 (2013), 561-576.
- [14] G. Perelman, The entropy formula for the Ricci flow and its geometric applications, *arXiv:math. DG/0211159*.
- [15] W.M. Sheng and X.J. Wang, Singularity profile in the mean curvature flow, *Methods Appl. Anal.* 16 (2009) no.2, 139-155.
- [16] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.* (2) 88 (1968), 62-105.
- [17] X.J. Wang, Convex solutions to the mean curvature flow, *Ann. of Math.* (2) 173 (2011) no.3, 1185-1239.
- [18] R.G. Ye, The logarithmic Sobolev inequality along the Ricci flow, *arXiv:0707.2424*.

Addresses:

Weimin Sheng: Department of Mathematics, Zhejiang University, Hangzhou 310027, China.

Haobin Yu: Department of mathematics, Zhejiang university, Hangzhou 310027, China.

Email: weimins@zju.edu.cn; robin1055@126.com.